

Feigen's Phenomenon in Terms of the Synthetic Theory

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Abstract The paper concerns with the analytic description of Feigen's results on plastic straining under combined loading. The model is developed in terms of the synthetic theory of irrecoverable deformation. Criterion to establish when Feigen's phenomenon can be observed is proposed. It is worthwhile to note that the synthetic theory requires no special additional assumptions, or generalizations to model Feigen's experiment.

Keywords: Feigen's phenomenon, plastic deformation, combined loading, synthetic theory of irrecoverable deformation

I. Introduction

Feigen's experimental results (Feigen, M., 1954) are of crucial importance to obtain further insight into mechanisms of plastic deforming. Despite the fact that this phenomenon dates back to the 1950s of the twentieth century, its modeling has not been completely solved yet.

Consider Feigen's experiment consisting in the following loading regime:

- (i) a specimen plastically deforms in a complex stress state, tension and torsion (Fig. 1a);
- (ii) the torsional component decreases and the tension increases, (Fig. 1b). The loading path (ABC , $AC > AB$) is shown in Fig. 2.

Feigen reports an unexpected phenomenon is observed. The torsional plastic deformation (γ^s) accumulated in the initial combined loading decreases during the torsional unloading, $\Delta\gamma^s < 0$ along BC portion: *the specimen is plastically "untwisting"*. According to classical ideas, a plastic strain developed in active loading remains unchangeable during the succeeding unloading.

This phenomenon can be explained by the fact that, along portion BC , plastic shifts develop within such slip systems that give negative increment in torsional strain thereby resulting in the plastic untwisting of specimen. At the same time, the fraction of grains, which produced positive torsional strains along portion AB , remains hardened along BC .

The aim of the paper is to model Feigen's experiments in terms of the synthetic theory of recoverable deformation. Another goal of the study is to establish a criterion for predicting the condition when the phenomenon is manifested. It is worth noting that the synthetic theory needs no additional assumptions or generalizations; its classical statement is readily applicable to the analytical description of Feigen's results. In contrast, for example, Mazilu, P. (1985) tried to explain this phenomenon by assuming the existence of certain hypoelastic deformations (Lehmann, Th., 1962) accompanying the elasto-plastic one.

The research presented here may be of interest for the studies dealing with a wide variety of engineering mechanics (Mirsalimov and Veliyev (2013); Szekeres (2012); Kharchenko and Sobkowski (2005); Zablocka et al. (2012); Malinin, V. (2011)).

It is clear that Feigen's phenomenon can not be modeled in terms of classical flow plasticity theories (for an extensive review, see Rees, 1987). If we take, for example, flow plasticity theory with isotropic hardening rule, it will relate plastic strain tensor components increment ($d\varepsilon_{ij}^s$) to stress deviator tensor components ($\bar{\sigma}_{ij}$) as

$$d\varepsilon_{ij}^s = \bar{\sigma}_{ij} C(\tau_0) d\tau_0, \quad (1.1)$$

where τ_0 is the second invariant of stress tensor and $C(\tau_0)$ is

$$C(\tau_0) = \frac{1}{2\tau_0} \left(\frac{1}{G_t} - \frac{1}{G} \right), \quad (1.2)$$

where G is elastic shear modulus and $G_t = d\tau_0/3d\gamma_0$ (γ_0 is the second invariant of strain tensor). Condition $G_t < G$ implies that $C(\tau_0) > 0$.

For the case of biaxial loading (tension + torsion), Eq. (1.1) give

$$d\varepsilon^S = \bar{\sigma}C(\tau_0)d\tau_0, \quad d\gamma^S = \tau C(\tau_0)d\tau_0. \quad (1.3)$$

In terms of the flow theory with isotropic hardening, plastic deformation is assumed to develop if $d\tau_0 > 0$. If to suggest that $d\tau_0 > 0$ along portion BC ($\bar{\sigma} > 0$ and $\tau > 0$), then Eq. (1.3) gives positive increments in both ε^S and γ^S components. For the case $d\tau_0 = 0$, we have $d\varepsilon^S = d\gamma^S = 0$. Both results contradict Feigen's experiment reporting that $d\gamma^S < 0$.

II. Mathematical apparatus: the synthetic theory of irrecoverable deformation.

The synthetic theory, which concerns with small strains of work-hardening metals, incorporates the elements of flow plasticity theory (Sanders, 1954) and slip concept (Budiansky, 1949). The fundamentals of the synthetic theory are put forth in details in Rusinko's works (2009, 2011).

(i) The synthetic theory belongs to two-level models. Similarly to the Budiansky slip concept, the synthetic theory has a micro- and a macro-level. For the macrolevel we take as the elementary volume of the body, \mathbb{V} , which is considered as point in the mathematical sense. The given volume, \mathbb{V} , consists of a large quantity of volumes, \mathbb{V}_0 (the micro-level), each being an element of the continuous, capable of deforming under the applied forces. Accordingly to Budiansky, it is assumed that the stress state in every volume \mathbb{V}_0 is the same as that in the volume \mathbb{V} . Plastic deformation in the microvolume \mathbb{V}_0 (grain) is assumed as slip of one part of \mathbb{V}_0 in relation to another. Therefore, in contrast to the even distribution of the stress above microvolumes \mathbb{V}_0 , the plastic deformation (or plastic slip) depends on the orientation of the slip system relative to the direction of the acting forces. It is assumed that the quantity of \mathbb{V}_0 is so great (theoretically it tends to infinity) that every possible orientation of slip systems exists in volume \mathbb{V} . The total deformation of \mathbb{V} is determined as the sum of the components of deformations generated within volumes \mathbb{V}_0 . Lichatchev et al. (1993) give the substantiation of the sizes of microvolumes \mathbb{V}_0 .

(ii) The establishment of stress~strain relationships on microlevel takes place in *the three-dimensional subspace, \mathbf{R}^3 , of the Ilyushin five-dimensional stress deviator space, \mathbf{R}^5* , where a loading is presented by *stress vector, $\vec{\mathbf{S}} \in \mathbf{R}^3$* , whose components are

$$S_1 = \sqrt{3/2}S_{xx}, \quad S_2 = S_{xx}/\sqrt{2} + \sqrt{2}S_{yy}, \quad S_3 = \sqrt{2}S_{xz}, \quad (2.1)$$

where S_{ij} are the stress deviator tensor components. The length of the vector $\vec{\mathbf{S}}$ is related to the second invariant of stress deviator tensor (J_2) as $|\vec{\mathbf{S}}| = 2\sqrt{3}J_2$.

(iii) New *yield criterion* is introduced, which coincides with nor Tresca's, no von-Mises's one in \mathbf{R}^5 . At the same time, its trace in \mathbf{R}^3 gives the von-Mises yield criterion:

$$S_1^2 + S_2^2 + S_3^2 = S_s^2, \quad S_s = \sqrt{2/3}\sigma_s \quad (2.2)$$

where σ_s is the yield strength of material in uniaxial tension. Through each point on the sphere we draw a tangent plane. So, the yield surface can be thought of the inner envelope of equidistant planes. The position of plane in \mathbf{R}^3 is defined by the following two quantities (Fig. 3):

a) Unit vector $\vec{\mathbf{m}}(m_1, m_2, m_3)$ normal to the plane:

$$m_1 = \cos\alpha \cos\beta, \quad m_2 = \sin\alpha \cos\beta, \quad m_3 = \sin\beta, \quad (2.3)$$

where α and β are spherical angles giving the orientation of $\vec{\mathbf{m}}$ in \mathbf{R}^3 (Rusinko, 2009). For simplicity, we consider only planes tangential to yield/loading surface in \mathbf{R}^3 and those tangential to the yield/loading surface in \mathbf{R}^5 are ignored.

b) The distance from the origin of coordinates to the plane, h_m .

The condition that a plane is located on the endpoint of stress vector is expressed as

$$h_m = \vec{\mathbf{S}} \cdot \vec{\mathbf{m}}. \quad (2.4)$$

(iv) According to Sanders, to establish a *hardening rule*, we extend the provision that a surface is constructed as an inner envelope of planes to the case of loading as well. During loading, the vector $\vec{\mathbf{S}}$ displaces on its endpoint a set of planes from their initial position, i.e. from sphere (2.2). Each plane moves without changing its

orientation. As a result, the loading surface – an inner envelope of planes – has a form as shown in Fig. 4. It consists of

- a cone whose lateral surface is formed by the boundary planes reached by the \vec{S} and
- the part of sphere (2.2), which is the inner envelope of stationary planes.

(v) Each tangent plane corresponds to an appropriate slip system, and the displacement of the plane on the endpoint of stress vector symbolizes the development of plastic microdeformation within the slip system. Plastic microstrain modeled by the displacement of one plane is assumed to be a vector normal to the plane (see Fig. 3). It is easy to see that the distance to a plane gives the degree of the hardening of material. Indeed, the greater h_m , the greater stress vector is needed to reach the plane, i.e. to induce plastic shift within the corresponding slip system. Further, we introduce an average measure of plastic strain developed within a slip system (i.e. for one plane), termed as plastic strain intensity (φ_m), as follows

$$\varphi_m = \frac{1}{r} \begin{cases} h_m^2 - S_s^2 & \text{if } h_m = \vec{S} \cdot \vec{m} \text{ (plane is reached by } \vec{S}) \\ 0 & \text{if } h_m < \vec{S} \cdot \vec{m} \text{ (plane is not reached by } \vec{S}) \end{cases} \quad (2.5)$$

where r is the constant of material, $[r] = \text{MPa}^2$. It must be noted that the product $\vec{S} \cdot \vec{m}$ is the resolve stress acting within a slip system. Further, it is easy to see that the greater h_m , the greater distance is traveled by the plane on the endpoint of stress vector, i.e. the greater plastic deformation develops within the corresponding slip system.

(vi) The macroplastic deformation, expressed via plastic strain vector components (e_k^S), is calculated as the sum of non-zero φ_m (Rusinko, 2009, 2011):

$$\vec{e}^S = \int d\vec{e}^S = \int \int \varphi_m \vec{m} dV, \text{ or } e_k^S = \int \int \varphi_m m_k dV, \quad dV = \cos \beta d\alpha d\beta, \quad k = 1, 2, 3. \quad (2.6)$$

The components e_k from Eq. (2.6) can be converted to the strain deviator tensor components (e_{ij}) as

$$e_1 = \sqrt{3/2} e_{xx}, \quad e_2 = e_{xx}/\sqrt{2} + \sqrt{2} e_{yy}, \quad e_3 = \sqrt{2} e_{xz} \quad (2.7)$$

It must be noted that, in terms of the synthetic theory, both loading and deformation are expressed via vectors: \vec{S} and \vec{e}^S , respectively. It is clear that vector algebra is much easier than tensor analysis. One more point that deserves attention is that vector components S_i have the same rights in the criterion (2.2); it is not the case if the von-Mises yield criterion is expressed via, e.g., stress deviator tensor components:

$$\left[S_{xx}^2 + S_{yy}^2 + S_{zz}^2 + 2(S_{xy}^2 + S_{yz}^2 + S_{xz}^2) \right]^{1/2} = 2\sqrt{3} J_2.$$

III. Isotropy postulate in terms of the synthetic theory

According to Ilyushin (1963), isotropy postulate reads that if the stress path in \mathbf{R}^3 is rotated in stress deviator space, then the corresponding strain path is rotated by the same amount (this postulate is valid for only von-Mises's medium). Consider an arbitrary loading path in \mathbf{R}^3 as shown in Fig. 5. Let the corresponding strain vector be \vec{e} , which makes angle η with the stress vector \vec{S} . Now, we rotate the loading path by a certain angle, δ . To demonstrate the fulfillment of the isotropy postulate, we rotate the coordinate system by the same angle δ . Within the rotated coordinate system, we obtain an analog of the previous loading and, therefore, it is easy to conclude that the angle between vectors \vec{e} and \vec{S} must be the same as in the initial case. It is clear that the value of strain strongly depends on the inner geometry of the loading path, but the rotation of loading path as a rigid figure does not affect the relation between vectors \vec{e} and \vec{S} at any point of loading path.

IV. Feigen's phenomenon in terms of synthetic theory

Relying upon isotropy postulate, we can model Feigen's phenomenon. We consider the following case:

- (i) an initial biaxial loading is proportional, i.e. the loading path of the stress vector \vec{S} is a straight line and (ii) after the loading an infinitesimal orthogonal additional loading is applied ($d\vec{S} \perp \vec{S}$). It is clear that such a loading regime falls within the scope of Feigen's experiment due to the additional loading gives an increase in tension and decrease in torsion. Let the vector \vec{S} make angle γ with S_1 -axis. Let us introduce a new system of

coordinates ($S_1' S_2' S_3'$) by rotating the initial system by the angle γ so that the stress vector is along the S_1' -axis. In fact, we obtain an analog of two-segment loading, uniaxial tension and additional torsion. Consequently, according to isotropy postulate, techniques developed in Rusinko's previous works (2009, 2011) can be readily used. In terms of the rotated coordinate system, the stress vector for the initial portion of loading has the only nonzero component:

$$S_1' = \sqrt{2/3}\sigma_x', \quad S_2' = 0 \quad S_3' = 0, \quad |\vec{S}| > S_S. \quad (4.1)$$

On the base of Eqs. (2.3)-(2.5), the distance to plane and strain intensity are

$$h_m' = S_1' m_1' = \sqrt{2/3}\sigma_x' \cos \alpha' \cos \beta', \quad (4.2)$$

$$\varphi_m' = \frac{2}{3r} [(\sigma_x')^2 \cos^2 \alpha' \cos^2 \beta' - \sigma_s^2], \quad (4.3)$$

where angles α' and β' are measured within $S_1' S_2' S_3'$ coordinate system in the same way as within the original system. The only non-zero component of plastic strain vector \vec{e}^S in $S_1' S_2' S_3'$ coordinate system is (by analogy with calculations from Rusinko's works (2009, 2011))

$$e_1^S = \iint_{\alpha' \beta'} \varphi_m' m_1' dV = \frac{2}{3r} \int_{-\alpha_1}^{\alpha_1} \int_{-\beta_1}^{\beta_1} [(\sigma_x')^2 \cos^2 \alpha' \cos^2 \beta' - \sigma_s^2] \cos \alpha' \cos^2 \beta' d\alpha' d\beta', \quad (4.4)$$

where

$$\cos \alpha_1(\beta') = \frac{b}{\cos \beta'}, \quad \cos \beta_1 = b, \quad b = \frac{\sigma_s}{\sigma_x'}. \quad (4.5)$$

By integrating (4.4) over domain (4.5) we obtain

$$e_1^S = \frac{4\sigma_s^2}{3r} \Phi(b), \quad \Phi(b) = \frac{1}{b^2} (1-b^2)^2, \quad \Phi(1) = 0. \quad (4.6)$$

Due to the action of $d\vec{S}$, which is analogous to additional torsion ($d\tau_{xz}'$) within $S_1' S_2' S_3'$ coordinate system, the strain intensity acquires the following increment

$$rd\varphi_m' = 2h_m' dh_m', \quad (4.7)$$

where

$$dh_m' = \tilde{h}_m' - h_m' = \sqrt{2} \sin \beta' d\tau_{xz}', \quad (4.8)$$

where \tilde{h}_m' stands for the distance to planes under the action of $\vec{S} + d\vec{S}$ vector:

$$\tilde{h}_m' = \sqrt{2/3}\sigma_x' \cos \alpha' \cos \beta' + \sqrt{2} \sin \beta' d\tau_{xz}'. \quad (4.9)$$

As follows from the analysis of two segment orthogonal loading (Rusinko, 2009), plastic strain vector increment components, de_1^S and de_3^S , are calculated as

$$de_1^S = \iint_{\alpha' \beta'} d\varphi_m' m_1' dV = \frac{4\sigma_x' d\tau_{xz}'}{\sqrt{3}r} \int_{-\alpha_1}^{\alpha_1} \int_0^{\beta_1} \cos^2 \alpha' \cos^3 \beta' \sin \beta' d\alpha' d\beta', \quad (4.10)$$

$$de_3^S = \iint_{\alpha' \beta'} d\varphi_m' m_3' dV = \frac{4\sigma_x' d\tau_{xz}'}{\sqrt{3}r} \int_{-\alpha_1}^{\alpha_1} \int_0^{\beta_1} \cos \alpha' \cos^2 \beta' \sin^2 \beta' d\alpha' d\beta', \quad (4.11)$$

Comparison of Eqs. (4.10) and (4.11) to Eq. (4.4) – β' -domain is $[0, \beta_1]$ and $[-\beta_1, \beta_1]$, respectively – shows that the set of planes being at the endpoint of \vec{S} vector is two times less than that due to the action of $\vec{S} + d\vec{S}$. Integrating in (4.10) and (4.11) yields

$$de_1^S = \frac{\sigma_s^2}{\sqrt{3}r\sigma_x'} \Phi(b) d\tau_{xz}', \quad \Phi(b) = \frac{1}{b^2} \arccos b + \frac{1}{b} \sqrt{1-b^2} (1-2b), \quad (4.12)$$

$$de_3^s = \frac{\sqrt{3}e_1^s}{2\sigma_x^s} d\tau_{xz}^s \quad (4.13)$$

The plot of $\Phi(b)$ is shown in Fig. 6. As we can see from the formulae above, the value of strain increments strongly depends on the value of initial loading.

The set of planes displaced by the vector $\bar{\mathbf{S}}$ is shown in Fig. 7a. For the case when the loading path of the vector $\bar{\mathbf{S}}$ is a straight line (proportional loading) (i) the normal vectors of displaced planes are symmetric above S_1 -axis, (ii) planes on either side of the $\bar{\mathbf{S}}$ incurs identical displacements. This means that the total plastic strain vector $\bar{\mathbf{e}}^s$, which is calculated as the sum of strain vectors, each of which is normal to the plane displaced by the $\bar{\mathbf{S}}$, lies along the S_1 -axis.

To model Feigen's phenomenon, one needs to analyze the sign of $\bar{\mathbf{e}}_3^s$ vector during additional loading. In Fig. 7b, planes with $m_3 > 0$ are depicted by solid lines; they give positive increments in $\bar{\mathbf{e}}_3^s$ caused by an orthogonal increment in stress. For planes depicted by dashed lines $m_3 < 0$, and, consequently, they stand with negative signs in the sum for the $\bar{\mathbf{e}}_3^s$. It is clear that the sign of $\bar{\mathbf{e}}_3^s$ depends on the set of "positive" and "negative" planes and the distances they traveled on the endpoint of stress vector.

The number of planes displaced by additional vector $d\bar{\mathbf{S}}$ is two times less than under the action of $\bar{\mathbf{S}}$ (Fig. 7b), and this set of the planes contains the same number of planes with negative normals (i.e. normals with negative components on S_3 -axis), while the number of planes with positive normals halves. This fact implies that the slope of the strain vector due to the action of $\bar{\mathbf{S}} + d\bar{\mathbf{S}}$ vector is less than that due to the $\bar{\mathbf{S}}$ vector. Feigen's phenomenon, a negative increment in plastic shear deformation during additional loading $d\bar{\mathbf{S}}$, is observed when the sum of plastic deformation increments having negative components on S_3 -axis is greater than that having positive components on S_3 -axis.

Now, we wish to establish a criterion for occurring of Feigen's phenomenon. In other words, we need to establish such an angle between stress vector $\bar{\mathbf{S}}$ and S_1 -axis, angle γ in Fig. 8, so that the orthogonal additional loading results in the negative component of $d\bar{\mathbf{e}}^s$ on S_3 -axis, $de_3^s < 0$. By projecting vector $d\bar{\mathbf{e}}^s = d\bar{\mathbf{e}}_1^s + d\bar{\mathbf{e}}_3^s$ on S_3 -axis and using Eqs. (4.12) and (4.13), we obtain

$$de_3^s = de_1^s \sin \gamma - de_3^s \cos \gamma \quad (4.14)$$

Eq. (4.14) at $de_3^s = 0$, together with (4.12) and (4.13), gives

$$\tan \gamma_0 = \frac{de_3^s}{de_1^s} = \frac{\frac{\sqrt{3}e_1^s}{2\sigma_x^s} d\tau_{xz}^s}{\frac{\sigma_s^2}{\sqrt{3}r\sigma_x^s} \Phi(b) d\tau_{xz}^s} = \frac{3re_1^s}{2\sigma_s^2 \Phi(b)} \quad (4.15)$$

If $\gamma < \gamma_0$, then $de_3^s < 0$, meaning that Feigen's phenomenon will be observed. This situation is shown in Fig. 8a. For the case $\gamma > \gamma_0$ (Fig. 8b), we have $de_3^s > 0$ implying a positive increment in plastic torsional deformation due to $d\bar{\mathbf{S}}$, which contradicts Feigen's phenomenon. Therefore, Eq. (4.15) can be treated as a criterion for occurring Feigen's phenomenon as a function of the parameters of initial loading.

V. Conclusion

The present paper has modeled Feigen's phenomenon in terms of the synthetic theory. A condition for the onset of negative shear plastic strain under the additional loading has been derived. We restrict ourselves to the case when the additional loading is perpendicular to the initial one. It has been shown that parameters of initial loading, the orientation and magnitude of stress vector, considerably affects the behavior of specimen under the second portion of loading.

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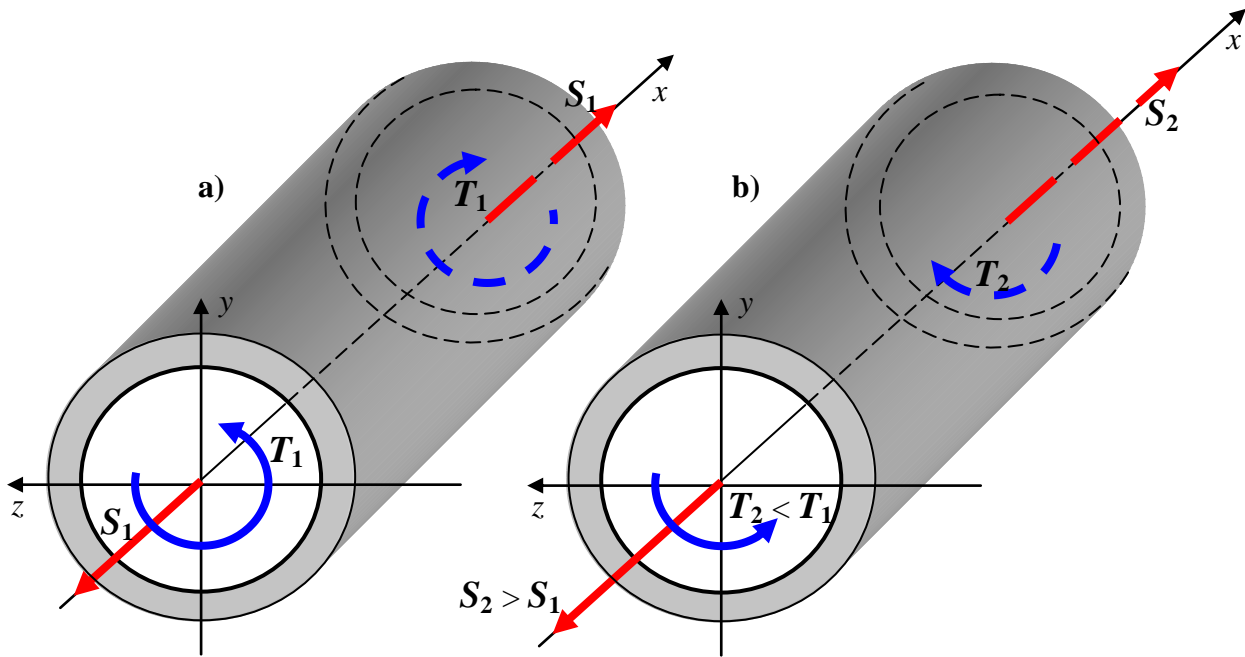


Fig. 1 Loading regime in Feigen's experiment; S denotes stretching force and T does torque

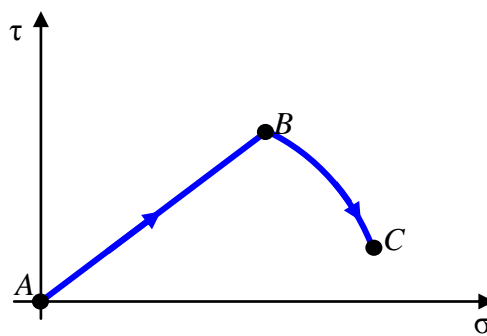


Fig. 2 Loading path in σ - τ plane in Feigen's experiment

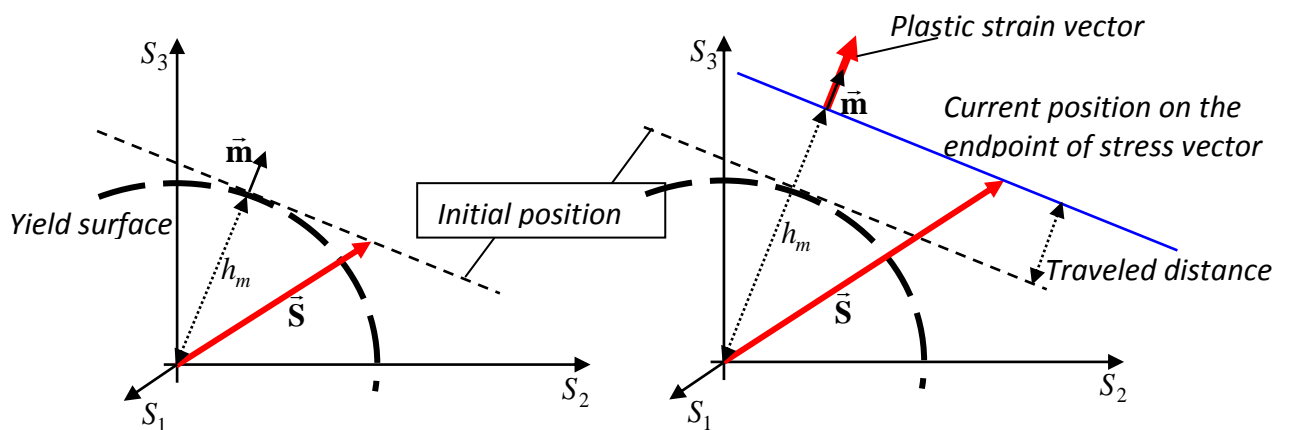


Fig. 3 Scheme of the displacement of tangent plane

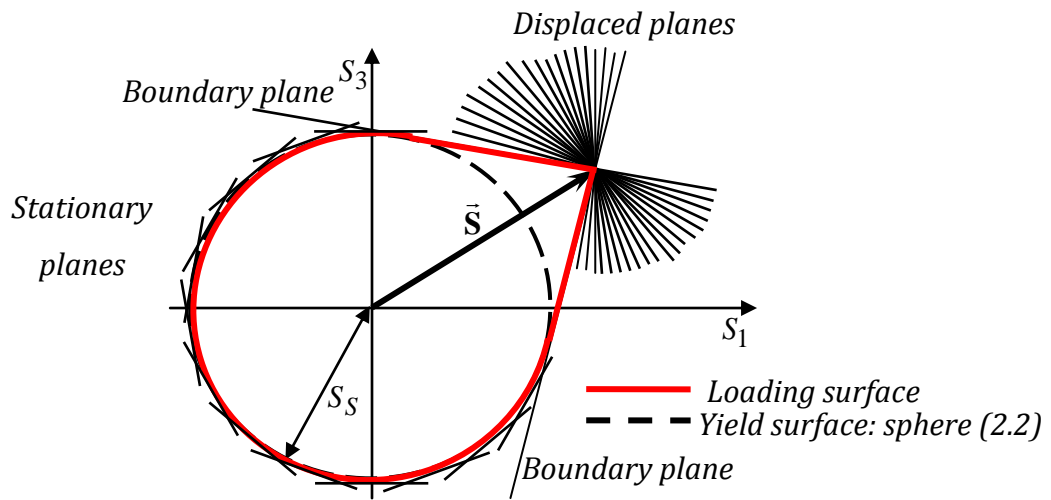


Fig. 4 Loading surface in terms of the synthetic theory

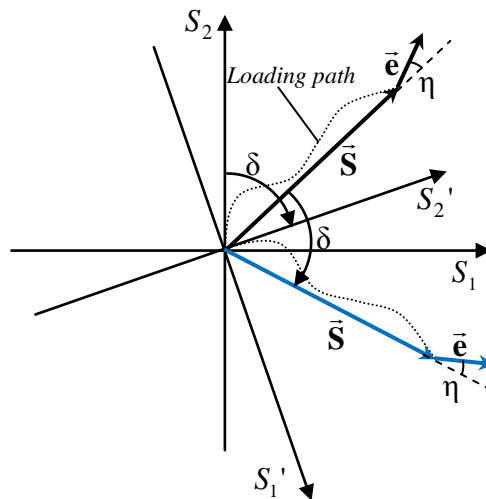


Fig. 5 On isotropy postulate

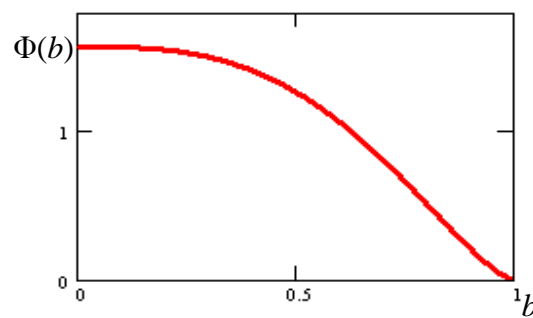


Fig. 6 Function $\Phi(b)$ from Eq. (4.12)

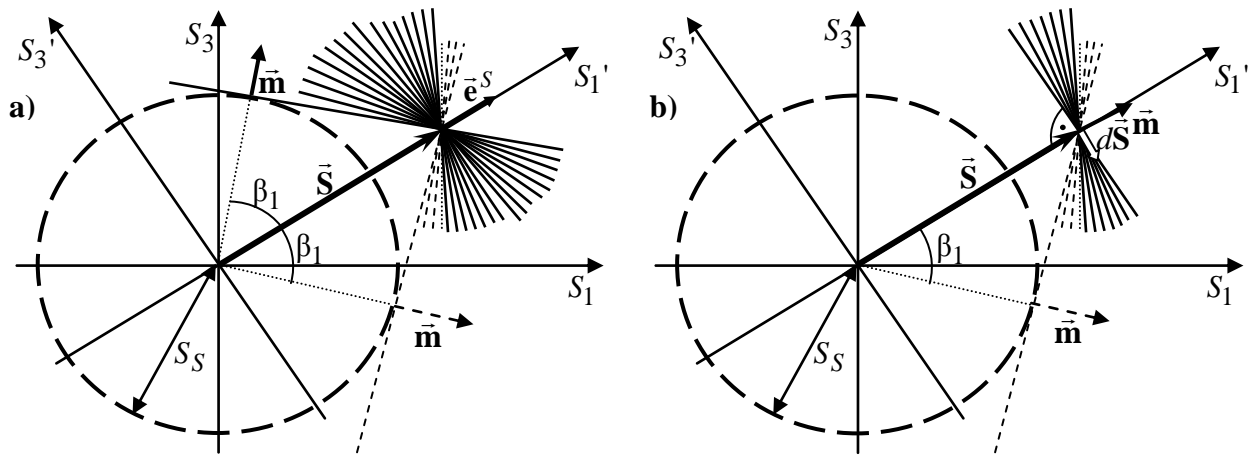


Fig. 7 Set of planes displaced by vector \bar{S} (a) and $d\bar{S}$ (b) in S_1-S_3 coordinate plane ($\alpha=0$).

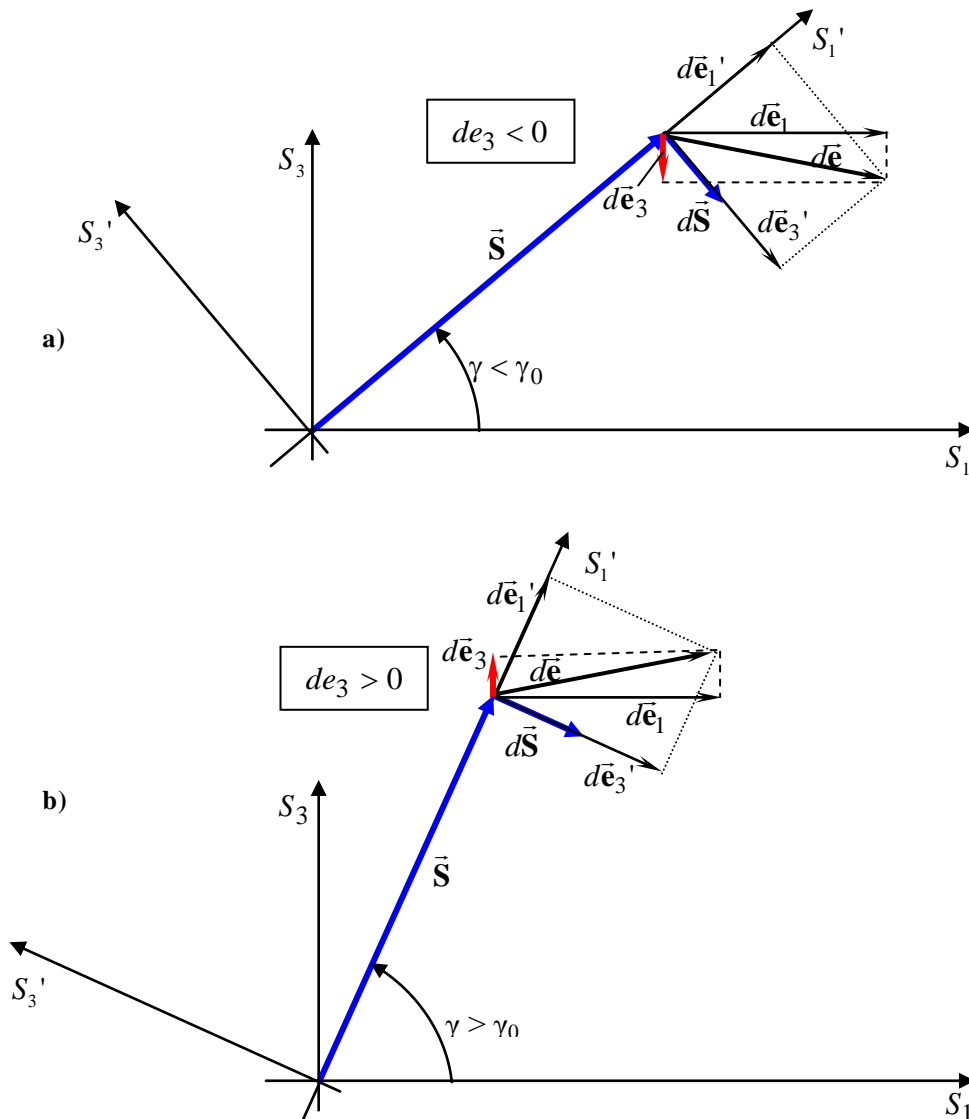


Fig. 8 The sign of de_3 in additional loading depending on the angle of initial loading (γ); superscripts “S” in strain vectors and components are avoided.